The Majorana representation of spins and the relation between $SU(\infty)$ and $SDiff(S^2)$

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ABSTRACT

The Majorana representation of spin-$\frac{n}{2}$ quantum states by sets of points on a sphere allows a realization of $SU(n)$ acting on such states, and thus a natural action on the two-dimensional sphere S^2. This action is discussed in the context of the proposed connection between $SU(\infty)$ and the group $SDiff(S^2)$ of area-preserving diffeomorphisms of the sphere. There is no need to work with a special basis of the Lie algebra of $SU(n)$, and there is a clear geometrical interpretation of the connection between the two groups. It is argued that they are *not* isomorphic, and comments are made concerning the validity of approximating groups of area-preserving diffeomorphisms by $SU(n)$.

I. INTRODUCTION

Groups of area-preserving diffeomorphisms and their Lie algebras have recently been the focus of much attention in the physics literature. Hoppe [1] has shown that in a suitable basis, the Lie algebra of the group $SDiff(S^2)$ of area-preserving diffeomorphisms of a sphere tends to that of $SU(N)$ as $N \to \infty$. This has obvious interest in connection with gauge theories of $SU(N)$ for large N. The use of $SU(N)$ for finite N as an approximation to the group of area-preserving diffeomorphisms has also been used in studies of supermembranes [2–4] and in particular has been used to argue for their instability.

The limiting procedure as $N \to \infty$ is delicate, and in particular, the need to take the limit in a particular basis makes one immediately wary as to how this result should be interpreted. In fact Hoppe and Schaller [5] have shown that there are infinitely many pairwise non-isomorphic Lie algebras, each of which tends to $su(\infty)$, the Lie algebra of $SU(\infty)$, as $N \to \infty$. The authors of references [3], [4] and [6,7] have especially emphasized the difficulties in relating such infinite limits with Lie algebras of area-preserving diffeomorphisms. Various authors have considered special limits and/or large-N limits of other classical Lie algebras [8–12] as relevant for 2-manifolds other than spheres. The purpose of this Letter is to clarify the geometrical nature of the relationship between $SU(N)$ and $SDiff(S^2)$.

In this Letter we will consider the group $SU(N)$ for $N \to \infty$ without the use of a specific basis for its Lie algebra, and in fact without consideration of its Lie algebra at all! The argument requires some familiarity with the Majorana representation of a spin-j system by a set of $2j$ points, not necessarily all distinct, on the surface of a 2-dimensional sphere, S^2.

II. THE MAJORANA REPRESENTATION OF SPIN

Note that a system with classical angular momentum $\vec{J}$ can be described by a single point on S^2 corresponding to the direction in which $\vec{J}$ points. The case in quantum mechanics is more subtle. A general state of spin j (in units of $\hbar$) must be represented by a collection of $2j$ points on the surface of a sphere, as first shown by Majorana [13] and later by Bacry [14] (for a summary of both papers, see reference [15]).

We reproduce the argument here for completeness, as it provides a connection between the action of $SU(2j)$ on the complex projective Hilbert space $\mathbb{C}^{2j}$ representing states of spin j and diffeomorphisms of S^2.

Let $\mathbb{CP}^{2j}$ denote the projective space associated with $\mathbb{C}^{2j+1}$. This is the space of $2j+1$-tuples in $\mathbb{C}^{2j+1}$ considered equivalent if they differ by a complex scale factor λ. That is, two points $(a_1, a_2, ..., a_{2j+1})$, and $(\lambda a_1, \lambda a_2, ..., \lambda a_{2j+1})$ of $\mathbb{C}^{2j+1}$ are considered the same point of $\mathbb{CP}^{2j}$.

Now consider the set P_{2j} of nonzero homogeneous polynomials of degree $2j$ in two complex variables, x, and y, which we associate to $(2j+1)$-tuples as follows:

$$(a_1, a_2, ..., a_{2j+1}) \to a_1 x^{2j} + a_2 x^{2j-1} y + \cdots + a_{2j+1} y^{2j} \tag{1}$$

Now (by the fundamental theorem of algebra) the polynomials can be factored into a product of $2j$ (not necessarily different) homogeneous terms linear in x and y. For each polynomial in P_{2j} we can associate an element of $\mathbb{C}^{2j+1}$ by writing it as a product of $2j$ factors:

$$a_1x^{2j} + a_2x^{2j-1}y + \cdots + a_{2j+1}y^{2j} = (\alpha_1x_1 - \beta_1y_1)(\alpha_2x_2 - \beta_2y_2)\cdots(\alpha_nx_n - \beta_{2j+1}y_{2j+1}) \quad (2)$$

for some complex α_i, β_i.

The coefficients of x and y in each term represent a point in $\mathbb{C}^{2j+1}$. This factorization is not unique, in that the terms can be permuted, and one can multiply any two factors by α and α^{-1} respectively. Thus we see that the whole space $\mathbb{CP}^{2j}$ in one-to-one correspondence with unordered sets of $2j$ points in $\mathbb{CP}^1$ corresponding to the $2j$ terms in equation (2) where we identify $(x_i, y_i) = \lambda(x_i, y_i)$ for any λ. But $\mathbb{CP}^1$ is S^2 and so we have the required result : *States of angular momentum $2j$ can be represented by single points in $\mathbb{CP}^{2j}$ or unordered sets of $2j$ (not necessarily distinct) points on S^2.*

Another, though less explicit, way to understand the Majorana representation is to note that a state of spin-j can be written as a totally symmetric product of $2j$ spin-1/2 wavefunctions.

III. $SU(N)$ AND ITS ACTION ON SPIN STATES

Recall that $\mathbb{CP}^n$ is the set of lines in $\mathbb{C}^{n+1}$ and can be written as the coset space

$$\mathbb{CP}^n = SU(n+1)/U(n) \quad (3)$$

Now for any coset space $S = G/H$, G acts transitively on S (that is, for any two points p and q of S, there is an element of G which takes p to q, so that $Gp = q$). Thus $SU(2n+1)$

has a natural action on $\mathbb{CP}^{2n}$, and thus on sets of points of S^2 such that any unordered set of points is carried to any other unordered set of points by a suitable transformation from $SU(2n+1)$.

So far, what has been presented is valid for any finite n. For each finite n then, we have a realization of the action of $SU(n)$ on S^2.

Now with this action on S^2, $SU(n)$ will *always* (even for finite n) contain transformations which carry two distinct points into the same point on S^2, and thus which will not correspond to diffeomorphisms of S^2. (We note in passing that $SDiff(S^2)$ is k-fold transitive for every positive integer k [16] where we recall that the action of a group G on a manifold M is said to be k-fold transitive if for any two arbitrary sets of k *distinct* points $(p_1, p_2, p_3, \ldots, p_k)$ and $(q_1, q_2, q_3, \ldots, q_k)$ of M there is an element of G which takes p_i to q_i for all $i = 1, \ldots, k$).

Thus in the limit of very large n, the permutations of sets of points on the sphere become much larger than the set of all diffeomorphisms of the sphere. In particular, any finite N approximation, $SU(N)$ of $SDiff(S^2)$ will contain mappings which do not correspond to elements of $SDiff(S^2)$. Thus we see that

$$\lim_{N\to\infty} SU(N) \not\simeq SDiff(S^2) \tag{4}$$

This can also be seen from another geometric view, where the Lie algebra of $SDiff(S^2)$ is that of divergenceless vector fields on S^2. Clearly, points pushed along these integral curves of these vector fields must not meet, and yet we see that there exist elements of $SU(N)$ acting on S^2 which will not satisfy this requirement. Of course one can argue that in some sense "most" of the transformations will not carry two distinct points into one, and

in fact this argument is implicit in the assumptions about the limiting procedures which associate the Lie algebras of $SDiff(M)$ and of $SU(\infty)$ for various choices of 2-manifold M [4], but this does not evade the fact that $SU(\infty)$ contains transformations which are clearly not in $SDiff(M)$.

It is interesting to note that $SU(\infty)$ as realized here, is, in fact, so large, and capable of such dramatic topology-changing distortions of S^2 that it may in fact be a useful description not of area-preserving diffeomorphisms of S^2, but rather of a wider class of deformations of S^2 including those which result in punctured spheres, 2-manifolds of different topologies, and 2-manifolds which have degenerated into 1-manifolds, or even a single point. Similar ideas have been put forth in [4].

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